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Bosbach states on L-graphs

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Abstract. In this article, we introduce Bosbach states on L-graphs. Firstly, the concept of Bosbach states on L-graphs is introduced. Then, some propositions of Bosbach states on L-graphs are proved. After that, some special Bosbach states on L-graphs are studied. In addition, some examples of Bosbach states on special L-graphs are provided. Moreover, we obtain that s is a Bosbach state on L-graph G' can not be characterized by s being a Bosbach state on L-graph G, where G' is the complement of L-graph G. On this basis, the concept of complement-preserving Bosbach states is proposed. Moreover, we obtain that in linear residuated lattices, G_1 and G_2 are isomorphic two L-graphs where h is a bijection from G_1 into G_2 and s is a Bosbach state on L-graph G_1, then s is also a Bosbach state induced by h on L-graph G_2; based on Example 3.21, if G_1 and G_2 are non-isomorphic two L-graphs and s is a Bosbach state on L-graph G_1, s may be also a Bosbach state on L-graph G_2. Finally, based on Example 3.23 and 3.24, we obtain that in residuated lattices, G_1 and G_2 are two isomorphic L-graphs, then s is a Bosbach state on L-graph G_2 can not be characterized by s being a Bosbach state on L-graph G_1.

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Keywords: Bosbach state; L-graph, Complement-preserving Bosbach state, Isomorphic.

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1. Introduction

Since Euler introduced graph theory in 1736 to solve the Konigsberg Bridge Problem, this mathematical discipline has evolved into a powerful tool for addressing complex real-world challenges across diverse fields (See [1, 2]). Graph theory has demonstrated remarkable versatility in modeling and solving practical problems in operations research, chemistry, computer science, and social sciences. Its applications range from urban transportation planning to cartographic precision in map-making. The continuous advancement of graph theory has further revealed its immense potential in tackling real-world problems. Each year brings new developments in the field (See [3, 4, 5, 6]), with many findings finding cross-disciplinary applications and contributing to solutions for some of humanity's most pressing challenges.

A seminal breakthrough occurred in 1965 when Zadeh [7, 8, 9] pioneered fuzzy set theory to address uncertainty and significant real-world phenomena. While Kaufman [10] first conceptualized fuzzy graphs as more realistic representations of natural systems, it was Rosenfeld [11] who established the theoretical foundation for their widespread applications across diverse domains including data mining, communications, clustering, scheduling theory, and planning. The emergence of fuzzy graph theory has spurred extensive research, evidenced by numerous publications in this field ([12, 13, 14, 15]). Its applications have expanded to critical areas such as cryptography and decision-making problems ([16, 17, 18, 19]). Notably, in 2022, Zahedi et al. [20] introduced the innovative concept of L-graphs (or RL graphs), investigating their fundamental properties and demonstrating practical utility.

The notion of states is an analogue to probability measure, and plays a very important role in the theory of quantum structures. In 1995, Mundici [21] introduced states on MV-algebras as averaging the truth value in Lukasiewicz logic. States constitute measures on their associated MV-algebras, which generalize the usual probability measures on Boolean algebras. Then, the notion of state has been extended to other logic algebras such as BL-algebras, residuated lattices, EQ-algebras,and their non-commutative cases. Different approaches to the generalization mainly give rise to two different notions, namely Riečan states and Bosbach states. In 2001, Dvurečenskij [22] proved a state on MV-algebras always exists. In 2004, Georgescu [23] defined Bosbach states and Riečcan states on pseudo BL-algebras, and for a good pseudo BL-algebra, he proved that any Bosbach state is also a Riečan state. He asked to find an example of Riečan state on a good pseudo BL-algebra which is not a Bosbach state. In 2017, Xin et al. [24] studied states on pseudo BCI-algebras. In 2020, Xin et al. [25] studied the notions of fantastic filters and investigated the existence of Bosbach states and Riečan states on EQ-algebras by using of fantastic filters. In 2021, Hua [26] studied states on L-algebras and derivations of L-algebras. In 2022, Shi et al. [27] investigated states on pseudo EQ-algebras and proved that any Bosbach state is a Riečan state in normal pseudo EQ-algebras, but the inverse is not true in general.

There is an arithmetic mean in addition and there is a geometric mean in multiplication on real number ring. Moreover, averaging the truth value in MV algebras is represented by states. In [20], Zahedi et al. introduced the innovative concept of L-graphs. L-graphs are defined based on residuated lattices. Unlike lattices, residuated lattices have an operation "$\rightarrow$". In L-graphs, to study the mean of the truth value of vertexes and edges, we need to use "$\rightarrow$" in residuated lattices. In L-graphs,the mean of the truth value of vertexes and edges is represented by Bosbach states in our paper. The purpose of studying Bosbach states on L-graphs in this article is to provide a tool for "fuzzy structural quantification analysis" on L-graphs.

In this article, we first attempt to introduce the notion of Bosbach states on L-graphs. After that, we will prove some propositions of Bosbach states on L-graphs.

Then, we will introduce some special Bosbach states and provide corresponding examples. In addition, some examples of Bosbach states on special L-graphs will be provided. Finally, we will discuss that (i) the relationship that between states on L-graph $G = (\alpha, \beta)$ and Bosbach states on the complement of L-graph $G' = (\alpha', \beta')$; (ii) the relationship between two L-graphs being isomorphic and Bosbach states on corresponding L-graphs.

2. Preliminaries

This section revisits fundamental definitions and properties of algebras pertinent to this paper.

Definition 2.1 ([28])**.** A path is a simple graph whose vertices can be ordered such that two vertices are adjacent iff they are consecutive in the list. In a path, if the first vertex and the last vertex are connected, we call it *cycle*.

Definition 2.2 ([28])**.** A graph G is said to be *connected*, if each pair of vertices in G belongs to a path.

Definition 2.3 ([28])**.** A *complete graph* G is a simple graph in which every pair of distinct vertices is connected by a unique edge.

Definition 2.4 ([28])**.** A graph $G = (V, E)$ is called *bipartite graph*, if V can be divided into two classes so that the vertices of any edge belong to different classes (vertices in the same class are not adjacent). In the bipartite graph, if any two vertices in different classes are connected, then we call such a graph a *complete bipartite graph*.

Definition 2.5 ([28])**.** A homomorphism from a simple graph $G = (V_G, E_G)$ to a simple graph $H = (V_H, E_H)$ is a surjection $f : V_G \to V_H$ such that $uv \in E_G$ iff $f(u)f(v) \in E_H$. f is called an *isomorphism*, if f is a homomorphism that is one-one.

Definition 2.6 ([29])**.** An algebra structure $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 1)$ is called a *residuated lattice*, if it satisfies the following conditions:

(R1) $(L, \wedge, \vee, , 0, 1)$ is a bounded lattice,

(R2) $(L, \odot, 1)$ is a commutative monoid (*i.e.* $\odot$ is commutative, associative and $x \odot 1 = x$ holds),

(R3) $x \odot y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in L$, where $\leq$ is the partial order of the lattice $(L, \wedge, \vee, 0, 1)$.

In what follows, by L we denote the universe of a residuated lattice $(L, \wedge, \vee, \odot, \to, 0, 1)$.

For any $x \in L$ and a natural number n, we define $x' = x \to 0$, $x'' = (x')'$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$, for $n \geq 1$.

Proposition 2.7 ([30])**.** *Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. Then for any $x, y, z \in L$, the following properties hold:*

(1) $1 \to x = x$, $x \to 1 = 1$,

(2) $x \leq y$ *if and only if* $x \to y = 1$,

(3) $x \odot x' = 0$, $x \odot y = 0$ *if and only if* $x \leq y'$,

(4) *if* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$ *and* $x \odot z \leq y \odot z$,

(5) $x \odot (x \to y) \le y$,
(6) $x \odot y \le x \wedge y \le x, y$ *and* $x \le y \to x$,
(7) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(8) $0' = 1$, $1' = 0$, $x \le x''$, $x''' = x'$,
(9) $x \odot (y \to z) \le y \to (x \odot z) \le (x \odot y) \to (x \odot z)$,
(10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(11) $x \vee (y \odot z) \ge (x \vee y) \odot (x \vee z)$, *hence* $x \vee y^n \ge (x \vee y)^n$ *and* $x^m \vee y^n \ge (x \vee y)^{mn}$ *for any natural numbers* m, n,
(12) $x \to (x \wedge y) = x \to y$,
(13) $x \odot y = x \odot (x \to x \odot y)$.

Definition 2.8 ([20])**.** $G = (\alpha, \beta)$ is called an *L-graph* on $G^* = (V, E)$, if $\alpha : V \to L$ and $\beta : E \to L$ are functions (L is a residuated lattice), with $\beta(q_i q_j) \le \alpha(q_i) \odot \alpha(q_j)$ for every $q_i q_j \in E$. Besides, if G^* is a path (cycle, bipartite, complete, complete bipartite) graph, then G is called a *path* (*cycle*, *bipartite*, *complete*, *complete bipartite*) *L-graph* on G^*.

Example 2.9 ([20])**.** Suppose $L = ([0, 1], \wedge, \vee, \odot, \to, 0, 1)$, where

$$a \odot b = \begin{cases} a + b - 1 & if\ a + b \ge 1 \\ 0 & if\ a + b < 1 \end{cases}$$

and

$$a \to b = \begin{cases} 1 & if\ b - a \ge 0 \\ 1 - a + b & if\ b - a < 0. \end{cases}$$

Then $G = (\alpha, \beta)$ is a cycle L-graph on G^*, as in Fig. 1, where $V = \{q_1, q_2, q_3, q_4\}$, $E = \{q_1 q_2, q_2 q_3, q_3 q_4, q_1 q_4\}$, $\beta(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$ for every $q_i q_j \in E$, $\alpha(q_1) = 0.8$, $\alpha(q_2) = 0.3$, $\alpha(q_3) = 0.1$, $\alpha(q_4) = 1$, $\beta(q_1 q_2) = 0.1$, $\beta(q_2 q_3) = 0$, $\beta(q_3 q_4) = 0.1$, $\beta(q_4 q_1) = 0.8$.

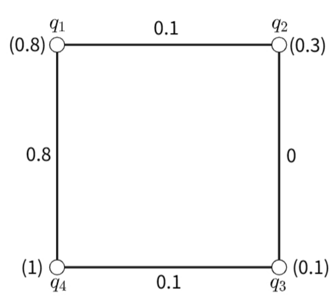

Figure 1. The L-graph G on G^*.

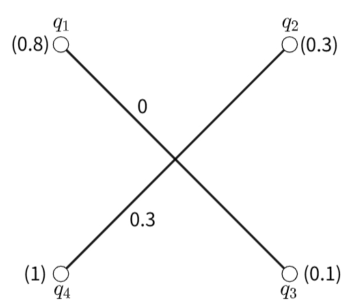

Figure 2. The complement of L-graph G'.

Definition 2.10 ([20])**.** The *complement* of L-graph G $= (\alpha, \beta)$ on $G^* = (V, E)$ is denoted by $G' = (\alpha', \beta')$ on $(G^*)' = (V, E')$ that is the complement of G^*, where $\alpha' = \alpha$, $\beta'(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$, for every $q_i q_j \notin E$.

Example 2.11 ([20])**.** Consider the L-graph G in Example 2.11. It is observed $G' = (\alpha', \beta')$ on $(G^*)'$, as in Fig. 2, where $V = \{q_1, q_2, q_3, q_4\}$, $E = \{q_1q_3, q_2q_4\}$, $\alpha'(q_1) = 0.8$, $\alpha'(q_2) = 0.3$, $\alpha'(q_3) = 0.1$, $\alpha'(q_4) = 1$, $\beta'(q_1q_3) = 0$ and $\beta'(q_2q_4) = 0.3$.

Definition 2.12 ([20])**.** Let $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ be two L-graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively and $c \in L \backslash \{1\}$. Then G_1 and G_2 are *isomorphic with threshold* c, denoted by $G_1 \cong G_2$, if there exists a bijection h: $V_1 \to V_2$ such that the following conditions hold: for all $q_1, q_2 \in V_1$ $(I1)$ $q_iq_j \in E_1$ if and only if $h(q_i)h(q_j) \in E_2$,

$(I2)$ $\alpha_1(q_i) > c$ if and only if $\alpha_2(h(q_i)) > c$,

$(I3)$ $\beta_1(q_iq_j) > c$ if and only if $\beta_2(h(q_i)h(q_j)) > c$.

Definition 2.13 ([31])**.** Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. A *Bosbach state* on L is a function $s : L \to [0, 1]$ such that the following conditions hold:

(i) $s(0) = 0$, $s(1) = 1$,

(ii) $s(x) + s(x \to y) = s(y) + s(y \to x)$ for all $x, y \in L$.

Definition 2.14 ([31])**.** Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. A *Riečan state* on L is a function $s : L \to [0, 1]$ such that the following conditions hold:

(i) $s(1) = 1$,

(ii) $s(x + y) = s(x) + s(y)$, whenever $x \perp y$.

3. Bosbach states on L-graphs

In the sections, we introduce the concept of Bosbach states on L-graphs, study some properties of Bosbach states on L-graphs provide some examples of Bosbach states on L-graphs and discuss the relationships between Bosbach states on L-graphs and L-graphs.

Definition 3.1. Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice, $G = (\alpha, \beta)$ be an L-graph on $G^* = (V, E)$, $q_i \& q_j$ be the edge formed by the vertices q_i and q_j, "$\cdot$" be multiplication in the real number ring and s: $L \to [0, 1]$ be a function satisfying the following condition: for all $q_i, q_j \in V$,

(BS1) $s(1) = 1$,

(BS2) $s(\alpha(q_i)) + s[\alpha(q_i) \to \alpha(q_j)] = s(\alpha(q_j)) + s[\alpha(q_j) \to \alpha(q_i)]$,

(BS3) $s(\beta(q_i \& q_j)) = s(\alpha(q_i)) \cdot s(\alpha(q_j))$.

Then s is said to be a *Bosbach state* on L-graphs.

Definition 3.2. Let s be a Bosbach state on L-graphs. Then the set

$$ker_V(s) = \{\alpha(q_i) \in L \mid s(\alpha(q_i)) = 1\}$$

is called the *kernel* on the vertexes V of a Bosbach state s on L-graphs and The set

$$ker_E(s) = \{\beta(q_i \& q_j) \in L \mid s(\beta(q_i \& q_j)) = 1\}$$

is called the *kernel* on the edges E of a Bosbach state s on L-graphs.

Proposition 3.3. *If $ker_E(s) \neq \varnothing$, then $ker_E(s) \subseteq ker_V(s)$.*

Proof. Based on Definition 3.2, since $ker_E(s) \neq \varnothing$, we have $s(\beta(q_i \& q_j)) = 1$. By Definition 3.1, we have $s(\alpha(q_i)) \cdot s(\alpha(q_j)) = 1$. Then $s(\alpha(q_i)) = s(\alpha(q_j)) = 1$. Thus $ker_E(s) \subseteq ker_V(s)$. □

Proposition 3.4. *Let s be a Bosbach state on L-graphs. If $s(0') = s(0)'$, then we have $s(0) = 0$.*

Proof. Based on Definition 2.6, Proposition 2.7 (8) and Definition 3.1, we have $1 = s(1) = s(0') = s(0)' = s(0) \to 0$. Then $s(0) \leq 0$. Thus we have $s(0) = 0$. □

Remark 3.5. Generally, $s(0) \neq 0$. We can check in Example 3.9.

Definition 3.6. Let s be a Bosbach state on L-graphs. If $s(0) = 0$, then s is said to be a *regular Bosbach state* on L-graphs.

Proposition 3.7. *Let s be a Bosbach state on L-graphs. If $\alpha(q_i) \leq \alpha(q_j)$, then $s(\alpha(q_i)) \leq s(\alpha(q_j))$.*

Proof. Since $\alpha(q_i) \leq \alpha(q_j)$. Then $\alpha(q_i) \to \alpha(q_j) = 1$ by Proposition 2.7 (2). Thus we have

$$s(\alpha(q_i)) + 1 = s(\alpha(q_i)) + s[\alpha(q_i) \to \alpha(q_j)] = s(\alpha(q_j)) + s[\alpha(q_j) \to \alpha(q_i)].$$

So $s(\alpha(q_i)) - s(\alpha(q_j)) = s[\alpha(q_j) \to \alpha(q_i)] - 1 \leq 0$. Hence $s(\alpha(q_i)) \leq s(\alpha(q_j))$. □

Definition 3.8. Let s be a Bosbach state on L-graphs. For any $q_i, q_j \in V$ and $1 \leq i < j$,

(i) if $\alpha(q_i) \leq \alpha(q_j)$, we have $s(\alpha(q_i)) \leq s(\alpha(q_j))$, then s is called an *increasing Bosbach state* on L-graphs,

(ii) if $\alpha(q_i) \leq \alpha(q_j)$, we have $s(\alpha(q_i)) \geq s(\alpha(q_j))$, then s is called a *decreasing Bosbach state* on L-graphs,

(iii) if $\alpha(q_i) = \alpha(q_j)$, we have $s(\alpha(q_i)) = s(\alpha(q_j))$, then s is called an *identity Bosbach state* on L-graphs.

Next, we will give some examples of Bosbach states on special L-graphs.

Example 3.9. Let $L = (\{0, a, b, c, 1\}, \odot, \to, \wedge, \vee)$, $0 < a < b < c < 1$. We define $\odot$ and $\to$ on L as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

For every $x, y \in L$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Then L is a residuated lattice. Consider the path L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 3, where $V = \{q_1, q_2\}$, $E = \{q_1 q_2\}$, $\alpha(q_1) = b$, $\alpha(q_2) = a$, $\beta(q_1 q_2) = 0$.

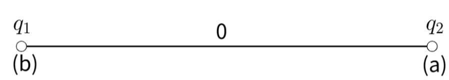

Figure 3. The path L-graph G on G^*.

Define the function $s : L \to [0, 1]$ by $s(1) = 1$, $s(0) = s(a) = 0$, $s(b) = 1$. Then s is a Bosbach state on the path L-graph $G = (\alpha, \beta)$ and s is also a regular Bosbach state on the path L-graph $G = (\alpha, \beta)$.

Example 3.10. Let $L = (\{0, a, b, c, d, 1\}, \odot, \rightarrow, \wedge, \vee)$, $0 \leq a \leq b \leq c \leq d \leq 1$. We define $\odot$ and $\rightarrow$ on L as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	c	c
d	0	0	0	b	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	d	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice. Consider the cycle L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 4, where $V = \{q_1, q_2, q_3\}$, $E = \{q_1q_2, q_2q_3, q_1q_3\}$, $\alpha(q_1) = b$, $\alpha(q_2) = c$, $\alpha(q_3) = c$, $\beta(q_1q_2) = a$, $\beta(q_2q_3) = c$, $\beta(q_3q_4) = a$.
Define the function $s : L \rightarrow [0, 1]$ by $s(1) = 1$, $s(0) = 0.64$, $s(a) = 0.4$, $s(b) = 0.5$, $s(c) = s(d) = 0.8$. Then s is a Bosbach state on the cycle L-graph $G = (\alpha, \beta)$ and s is also an increasing Bosbach state on the cycle L-graph $G = (\alpha, \beta)$.

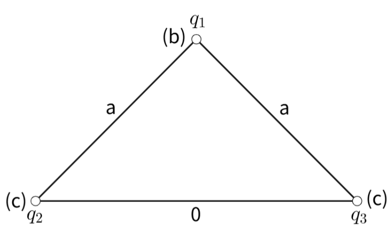

Figure 4. The cycle L-graph G on G^*.

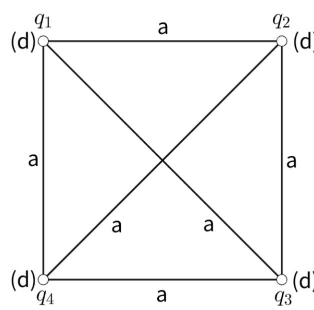

Figure 5. The complete L-graph G on G^*.

Example 3.11. Consider the residuated lattice in Example 3.10 and the complete L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 5, where $V = \{q_1, q_2, q_3, q_4\}$, $E = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3, q_2q_4\}$, $\alpha(q_1) = d$, $\alpha(q_2) = d$, $\alpha(q_3) = d$, $\alpha(q_4) = d$, $\beta(q_1q_2) = a$, $\beta(q_2q_3) = a$, $\beta(q_3q_4) = a$, $\beta(q_4q_1) = a$, $\beta(q_1q_3) = a$, $\beta(q_2q_4) = a$.
Define the function $s : L \rightarrow [0, 1]$ by $s(1) = 1$, $s(0) = 0$, $s(a) = 0.49$, $s(b) = 0.2$, $s(c) = 0.6$, $s(d) = 0.7$. Then s is a Bosbach state on the complete L-graph $G = (\alpha, \beta)$ and s is also an identity Bosbach state on the complete L-graph $G = (\alpha, \beta)$.

Example 3.12. Let $L = (\{0, a, b, c, d, 1\}, \odot, \rightarrow, \wedge, \vee)$, $0 \leq a, b < c < d < 1$, where a and b are incomparable. We define $\odot$ and $\rightarrow$ on L as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice. Consider the bipartite L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 6, where $V = \{q_1, q_2, q_3, q_4, q_5\}$, $E = \{q_1q_3, q_1q_5, q_2q_4\}$, $\alpha(q_1) = d$, $\alpha(q_2) = d$, $\alpha(q_3) = 1$, $\alpha(q_4) = 1$, $\alpha(q_5) = 1$, $\beta(q_1q_3) = c$, $\beta(q_1q_5) = b$, $\beta(q_2q_4) = a$. Define the function $s : L \to [0, 1]$ by $s(0) = 0$,$s(1) = 1$, $s(a) = s(b) = s(c) = s(d) = 0.6$. Then s is a Bosbach state on the bipartite L-graph $G = (\alpha, \beta)$.

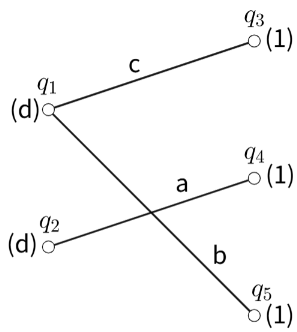

Figure 6. The bipartite L-graph G on G^*.

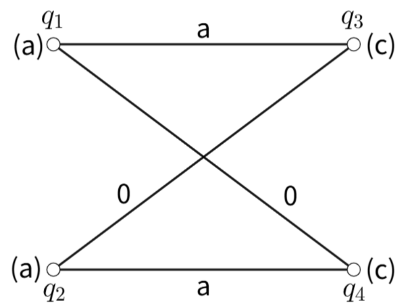

Figure 7. The complete bipartite L-graph G on G^*.

Example 3.13. Consider the residuated lattice in Example 3.12 and the complete bipartite L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 7, where $V = \{q_1, q_2, q_3, q_4\}$, $E = \{q_1q_3, q_1q_4, q_2q_3, q_2q_4\}$, $\alpha(q_1) = a$, $\alpha(q_2) = a$, $\alpha(q_3) = c$, $\alpha(q_4) = c$, $\beta(q_1q_3) = a$, $\beta(q_1q_4) = 0$, $\beta(q_2q_3) = a$, $\beta(q_2q_4) = 0$.
Define the function $s : L \to [0, 1]$ by $s(0) = s(a) = s(c) = s(1) = 1$, $s(b) = 0.2$, $s(d) = 0.7$. Then s is a Bosbach state on the complete bipartite L-graph $G = (\alpha, \beta)$ and s is also a decreasing Bosbach state on L-graphs $G = (\alpha, \beta)$.

In the following section, we will discuss the relationship between Bosbach states on L-graphs and Bosbach states on the complement of L-graphs.

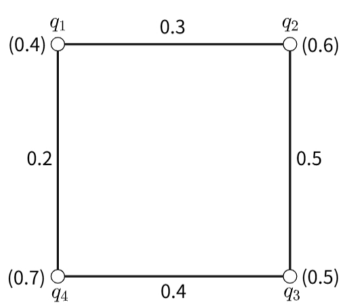

Figure 8. The L-graph G on G^*.

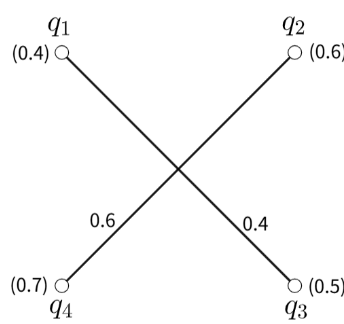

Figure 9. The complement of L-graph G' on $(G^*)'$.

Example 3.14. Let $L = ([0,1], \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice, where

$$a \odot b = a \wedge b \text{ and } a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

Consider the cycle L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 8, where $V = \{q_1, q_2, q_3, q_4\}$, $E = \{q_1q_2, q_2q_3, q_3q_4, q_1q_4\}$, $\alpha(q_1) = 0.4$, $\alpha(q_2) = 0.6$, $\alpha(q_3) = 0.5$, $\alpha(q_4) = 0.7$, $\beta(q_1q_2) = 0.3$, $\beta(q_2q_3) = 0.5$, $\beta(q_3q_4) = 0.4$, $\beta(q_4q_1) = 0.2$.
Now, we define the function s: $L \rightarrow [0,1]$ as follows:

$$s(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.1 \\ 1 & \text{if } 0.1 < x \leq 1. \end{cases}$$

One can easily check that s is a Bosbach state on L-graph $G = (\alpha, \beta)$.
Consider the cycle L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$. It is observed $G' = (\alpha', \beta')$ on $(G^*)' = (V, E')$, as in Fig. 9, where $V = \{q_1, q_2, q_3, q_4\}$, $E' = \{q_1q_3, q_2q_4\}$, $\alpha'(q_1) = 0.4$, $\alpha'(q_2) = 0.6$, $\alpha'(q_3) = 0.5$, $\alpha'(q_4) = 0.7$, $\beta'(q_1q_3) = 0.4$, $\beta'(q_2q_4) = 0.6$. Consider s on L-graph G. It is clear that s is a Bosbach state on L-graph $G' = (\alpha', \beta')$.

Example 3.15. Let $L = ([0,1], \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice, where

$$a \odot b = \begin{cases} a + b - 1 & \text{if } a + b \geq 1 \\ 0 & \text{if } a + b < 1 \end{cases}$$

and

$$a \rightarrow b = \begin{cases} 1 & \text{if } b - a \geq 0 \\ 1 - a + b & \text{if } b - a < 0. \end{cases}$$

Consider the L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 10, where $V = \{q_1, q_2, q_3\}$, $E = \{q_1q_2, q_1q_3\}$, $\alpha(q_1) = 0.5$, $\alpha(q_2) = 0.6$, $\alpha(q_3) = 0.5$, $\beta(q_1q_2) = 0.4$, $\beta(q_1q_3) = 0.3$.
Now, we define the function s: $L \rightarrow [0,1]$ as follows:

$$s(x) = \begin{cases} 0.25 & \text{if } 0 \leq x \leq 0.4 \\ 0.5 & \text{if } 0.4 < x \leq 0.6 \\ 1 & \text{if } 0.6 < x \leq 1. \end{cases}$$

One can easily check that s is a Bosbach state on L-graph $G = (\alpha, \beta)$.

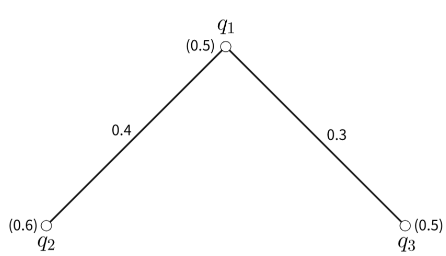

Figure 10. The L-graph G on G^*.

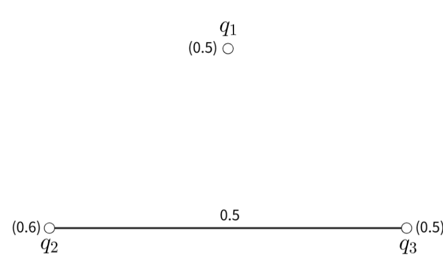

Figure 11. The complement of L-graph G' on $(G^*)'$.

Consider the L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$. It is observed $G' = (\alpha', \beta')$ on $(G^*)' = (V, E')$, as in Fig. 11, where $V = \{q_1, q_2, q_3\}$, $E' = \{q_2q_3\}$, $\alpha'(q_1) = 0.5$, $\alpha'(q_2) = 0.6$, $\alpha'(q_3) = 0.5$, $\beta'(q_2q_3) = 0.5$. Consider s on L-graph G. Since $s(\beta'(q_2\&q_3)) = s(0.5) = 0.5$, $s(\alpha'(q_2)) = s(0.6) = 0.5$, $s(\alpha'(q_3)) = s(0.5) = 0.5$, $s(\alpha'(q_2)) \cdot s(\alpha'(q_3)) = 0.5 \times 0.5 = 0.25 \neq 0.5$. Hence, s is not a Bosbach state on L-graph $G' = (\alpha', \beta')$.

Remark 3.16. The above two examples express that s is a Bosbach state on L-graph $G' = (\alpha', \beta')$ can not be characterized by s being a Bosbach state on L-graph $G = (\alpha, \beta)$.

Definition 3.17. If s is a Bosbach state on L-graphs and s is also a Bosbach state on the complement of L-graphs, then we called s is a *complement-preserving Bosbach state*.

It is clear that s is a complement-preserving Bosbach state in Example 3.14.

Proposition 3.18. *Let L-graph $G'' = (\alpha'', \beta'')$ be the complement of L-graph $G' = (\alpha', \beta')$ and $G' = (\alpha', \beta')$ be the complement of L-graph $G = (\alpha, \beta)$. If s is a complement-preserving Bosbach state and $\beta(q_i\&q_j) = \alpha(q_i) \odot \alpha(q_j)$, then s is also a Bosbach state on $G'' = (\alpha'', \beta'')$.*

Proof. Based on Definition 2.8 and Definition 2.10, in L-graph $G'' = (\alpha'', \beta'')$ and L-graph $G = (\alpha, \beta)$, we have $\alpha(q_i) = \alpha''(q_i)$ and $\beta''(q_i\&q_j) = \alpha''(q_i) \odot \alpha''(q_j)$ for every $q_i, q_j \in V$. Since $\beta(q_i\&q_j) = \alpha(q_i) \odot \alpha(q_j)$, we have $\beta''(q_i\&q_j) = \beta(q_i\&q_j)$. Since s a complement-preserving Bosbach state. Then s is a Bosbach state on $G'' = (\alpha'', \beta'')$ by Definition 3.1. □

In this section, we will discuss the relationship between two L-graphs being isomorphic and Bosbach states on corresponding L-graphs.

Theorem 3.19. *Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a linear residuated lattice, $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ be two L-graphs and s be a Bosbach state on L-graph $G_1 = (\alpha_1, \beta_1)$. If G_1 and G_2 are isomorphic two L-graphs where h is a bijection from G_1 into G_2, then s is also a Bosbach state induced by h on L-graph $G_2 = (\alpha_2, \beta_2)$.*

Proof. Based on Definition 2.12 and Definition 3.1, it is clear that $s(1) = 1$, $s(\alpha_2(q'_i)) + s[\alpha_2(q'_i) \rightarrow \alpha_2(q'_j)] = s(\alpha_2(q'_j)) + s[\alpha_2(q'_j)) \rightarrow \alpha_2(q'_i)]$ and $s(\beta_2(q'_i\&q'_j)) = s(\alpha_2(q'_i)) \cdot s(\alpha_2(q'_j))$. Then s is a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$. □

Example 3.20. Let $L = ([0, 1], \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice, where

$$a \odot b = a \wedge b \text{ and } a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

Consider two L-graphs $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, as in Fig. 12, where $V_1 = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3\}$, $\alpha_1(q_1) = 0.3$, $\alpha_1(q_2) = 0.5$, $\alpha_1(q_3) = 0.6$, $\alpha_1(q_4) = 0.8$, $\beta_1(q_1q_2) = 0.2$, $\beta_1(q_2q_3) = 0.4$, $\beta_1(q_3q_4) = 0.6$, $\beta_1(q_4q_1) = 0.3$, $\beta_1(q_1q_3) = 0.1$, $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3\}$, $\alpha_2(q'_1) = 0.3$, $\alpha_2(q'_2) = 0.5$,

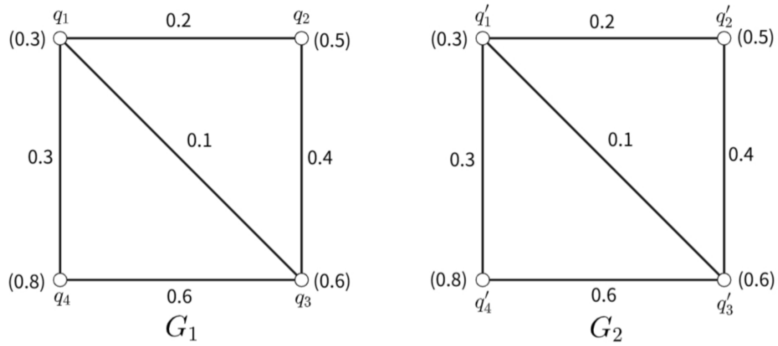

Figure 12. Two L-graphs G_1 and G_2 .

$\alpha_2(q'_3) = 0.6$, $\alpha_2(q'_4) = 0.8$, $\beta_2(q'_1q'_2) = 0.2$, $\beta_2(q'_2q'_3) = 0.4$, $\beta_2(q'_3q'_4) = 0.6$, $\beta_2(q'_4q'_1) = 0.3$, $\beta_2(q'_1q'_3) = 0.1$. Additionally, let h be a bijection from G_1 into G_2, where $h(q_i) = q'_i$. Therefore, G_1 and G_2 are isomorphic L-graphs.
Now, we define the function s: $L \to [0,1]$ as follows:

$$s(x) = \begin{cases} 0.3 & \text{if } 0 \leq x \leq 0.3 \\ 1 & \text{if } 0.3 < x \leq 1. \end{cases}$$

One can easily check that s is a Bosbach state on L-graph $G_1 = (\alpha_1, \beta_1)$. It is clear that s is also a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$.

Example 3.21. Let $L = ([0,1], \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice, where

$$a \odot b = a \wedge b \text{ and } a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

Consider two L-graphs $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, as in Fig. 13, where $V_1 = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3\}$, $\alpha_1(q_1) = 0.8$, $\alpha_1(q_2) = 0.4$, $\alpha_1(q_3) = 0.9$, $\alpha_1(q_4) = 0.6$, $\beta_1(q_1q_2) = 0.4$, $\beta_1(q_2q_3) = 0.3$, $\beta_1(q_3q_4) = 0.5$, $\beta_1(q_4q_1) = 0.6$, $\beta_1(q_1q_3) = 0.7$, $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3\}$, $\alpha_2(q'_1) = 0.7$, $\alpha_2(q'_2) = 0.4$, $\alpha_2(q'_3) = 0.8$, $\alpha_2(q'_4) = 0.6$, $\beta_2(q'_1q'_2) = 0.3$, $\beta_2(q'_2q'_3) = 0.4$, $\beta_2(q'_3q'_4) = 0.6$, $\beta_2(q'_4q'_1) = 0.5$, $\beta_2(q'_1q'_3) = 0.7$. Additionally, let h be a bijection from G_1 into G_2, where $h(q_i) = q'_i$. Then G_1 and G_2 are non-isomorphic L-graphs.
Now, we define the function s: $L \to [0,1]$ as follows:

$$s(x) = \begin{cases} 0.4 & \text{if } 0 \leq x \leq 0.4 \\ 1 & \text{if } 0.4 < x \leq 1. \end{cases}$$

One can easily check that s is a Bosbach state on L-graph $G_1 = (\alpha_1, \beta_1)$. It is clear that s is also a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$.

Remark 3.22. Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a linear residuated lattice, G_1 and G_2 are two non-isomorphic L-graphs. The above example expresses that if s is a Bosbach state on L-graph $G_1 = (\alpha_1, \beta_1)$, s may be also a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$.

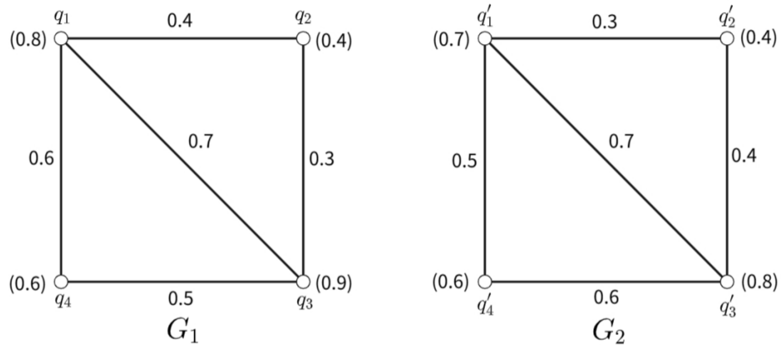

Figure 13. Two L-graphs G_1 and G_2 .

Example 3.23. Let $L = (\{0, a, b, c, d, e, 1\}, \odot, \to, \wedge, \vee)$, $0 < a < b, c < d < e < 1$, where b and c are incomparable. We define $\odot$ and $\to$ on L as follows:

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	c	c	c	c
d	0	a	a	c	c	c	d
e	0	a	b	c	d	e	e
1	0	a	b	c	d	e	1

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	d	1	d	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	d	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

Then L is a residuated lattice. Consider two L-graphs $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, as in Fig. 14, where $V_1 = \{q_1, q_2\}$, $E_1 = \{q_1q_2\}$, $\alpha_1(q_1) = b$, $\alpha_1(q_2) = e$, $\beta_1(q_1q_2) = a$, $V_2 = \{q_1', q_2'\}$, $E_2 = \{q_1'q_2'\}$, $\alpha_2(q_1') = c$, $\alpha_2(q_2') = e$, $\beta_2(q_1'q_2') = a$. Additionally, let h be a bijection from G_1 into G_2, where $h(q_i) = q_i'$. Then G_1 and G_2 are isomorphic L-graphs.

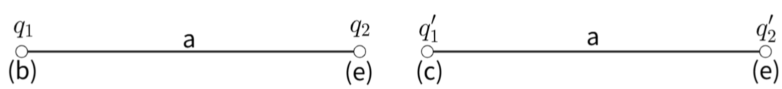

Figure 14. Two L-graphs G_1 and G_2 .

Define the function $s : L \to [0, 1]$ by $s(0) = 0$, $s(a) = s(b) = s(c) = s(d) = 0.6$, $s(1) = 1$. One can easily check that s is a Bosbach state on L-graph $G_1 = (\alpha_1, \beta_1)$. It is clear that s is also a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$.

Example 3.24. Consider the residuated lattice in Example 3.23. Consider two L-graphs $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, as in Fig. 15, where $V_1 = \{q_1, q_2, q_3\}$, $E_1 = \{q_1q_2, q_2q_3, q_1q_3\}$, $\alpha_1(q_1) = e$, $\alpha_1(q_2) = d$, $\alpha_1(q_3) = d$, $\beta_1(q_1q_2) = b$, $\beta_1(q_2q_3) = a$, $\beta_1(q_1q_3) = b$, $V_2 = \{q_1', q_2', q_3'\}$, $E_2 = \{q_1'q_2', q_2'q_3', q_1'q_3'\}$, $\alpha_2(q_1') = e$, $\alpha_2(q_2') = d$, $\alpha_2(q_3') = d$, $\beta_2(q_1'q_2') = c$, $\beta_2(q_2'q_3') = a$, $\beta_2(q_1'q_3') = c$. Additionally, let h be a bijection from G_1 into G_2, where $h(q_i) = q_i'$. Then G_1 and G_2 are isomorphic L-graphs.

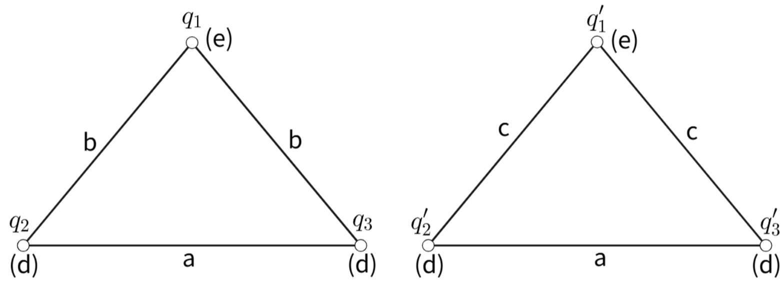

Figure 15. Two L-graphs G_1 and G_2 .

Define the function $s : L \to [0,1]$ by $s(0) = 0.2$, $s(a) = 0.25$, $s(b) = s(d) = 0.5$, $s(c) = 0.4$, $s(1) = 1$. One can easily check that s is a Bosbach state on L-graph $G_1 = (\alpha_1, \beta_1)$. Since $s(\beta_2(q_1' \& q_2')) = s(c) = 0.4$, $s(\alpha_2(q_1')) = s(e) = 1$, $s(\alpha_2(q_2')) = s(d) = 0.5$, $s(\alpha_2(q_1')) \cdot s(\alpha_2(q_2')) = 1 \times 0.5 = 0.5 \neq 0.4$. Thus s is not a a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$.

Remark 3.25. Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice, $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ be two isomorphic L-graphs. The above two examples express that s is a Bosbach state on L-graph $G_2 = (\alpha_2, \beta_2)$ can not be characterized by s being a Bosbach state on on L-graph $G_1 = (\alpha_1, \beta_1)$.

4. Conclusion

In this paper, we first introduced the notion of Bosbach states on L-graphs. Then, some propositions of Bosbach states on L-graphs were proved. After that, some special Bosbach states on L-graphs were proposed. Additionally, we introduced some examples of Bosbach states on special L-graphs. Moreover, we obtained that s is a Bosbach state on L-graph $G' = (\alpha', \beta')$ can not be characterized by s being a Bosbach state on L-graph $G = (\alpha, \beta)$ and the concept of complement-preserving Bosbach states on L-graphs was introduced. Finally, we obtained that the relationship between two L-graphs being isomorphic and Bosbach states on corresponding L-graphs.

In the future, we will study Bosbach states on more complex L-graphs and attempt to obtain some other properties of Bosbach states on L-graphs.

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References

[1] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno and C. Zhou, Synchronization in complex networks, Physics Reports 469 (2008) 93–153.
[2] A. Barrat, M. Barth'elemy and A. Vespignani, Dynamical processes in complex networks, UK: Cambridge University Press 2008.
[3] S. Banitalebi and R. A. Borzooei, Neutrosophic special dominating set in neutrosophic graphs, Neutrosophic Sets and Systems 45 (2021) 26–39.

[4] S. Banitalebi and R. A. Borzooei, Domination of vague graphs by using of strong arcs, Journal of Mathematical Extention 16 (2022) 1–22.
[5] R. A. Borzooei and M. Shadravan, On l-reconstructibility of degree list of graphs, AUT Journal of Mathematics and Computing 5 (2024) 39–44.
[6] M. Shadravan and R. A. Borzooei, Coupon coloring of Kneser graph $K(n, 2)$, Discrete Mathematics, Algorithms and Applications 16 (2024) 2350020.
[7] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[8] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[9] L. A. Zadeh and R. A. Aliev, Fuzzy logic theory and applications, part I and part II sets, New jersey: World scientific publishing 2018.
[10] A. Kauffman, Introduction a La theorie des sousensembles flous: A L'Usage Des Ingenieurs (Fuzzy Sets Theory), Tome III, Masson Et Cie, Editeurs, French 1975.
[11] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appli. 35 (1971) 512–517.
[12] S. Banitalebi and R. A. Borzooei, Domination in Pythagorean fuzzy graphs, Granular Computing (2023) 959–966.
[13] S. Banitalebi, S. S. Ahn and R. A. Borzooei, Special irregular neutrosophic graphs, Journal of Intelligent and Fuzzy Systems 46 (2024) 4147–4157.
[14] R. A. Borzooei, B. Sheikh Hoseini and M. Mohseni Takallo, Results on t-fuzzy graphs, New Mathematics and Natural Computation 16 (2020) 143–161.
[15] M. Khalili and R. A. Borzooei, Notes on energy of matching in fuzzy graphs, 2022 9th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS), Bam, Iran 2022 1–4.
[16] M. Khalili and R. A. Borzooei, Matching in n-th Type intuitionistic fuzzy graphs, 2020 8th Iranian Joint Congress on Fuzzy and intelligent Systems (CFIS) 80–85.
[17] M. Khalili and R. A. Borzooei, Matching numbers in fuzzy graphs, Journal of Applied Mathematics and Computing 67 (2021) 1–22.
[18] M. Shamsizadeh, M. M. Zahedi, M. Golmohamadian and Kh. Abolpour, Zero-forcing Finite Automata, International Journal of Industrial Mathematics 13 (2021) 477–488.
[19] M. Shamsizadeh, M. M Zahedi, Kh. Abolpour and M. De la Sen, Recognizable Languages of k-Forcing Automata, Mathematical and Computational Applications 29 (2024) 32.
[20] E. Raisi Sarbizhan, M. M. Zahedi and M. Shamsizadeh, L-graph automata and some applications, The Computer Journal 66 (2023) 1698–1716.
[21] D. Mundici, Averaging the truth-value in Lukasiewicz logic, Studia Logica 55 (1995) 113–127.
[22] A. Dvurečenskij, States on pseudo MV-algebras, Studia logica 68 (2001) 301–327.
[23] G. Georgescu, Bosbach states on fuzzy structures, Soft Computing 8 (2004) 217–230.
[24] X. L. Xin, Y. J. Li and Y. L. Fu, States on pseudo-BCI algebras, European Journal of Pure and Applied Mathematics 10 (2017) 455–472.
[25] X. L. Xin, Y. C. Ma and Y. L. Fu, The existence of states on EQ-algebras, Mathematica Slovaca 70 (2020) 527–546.
[26] X. J. Hua, State L-algebras and derivations of L-algebras, Soft Computing 25 (2021) 4201–4212.
[27] J. Q. Shi, X. L. Xin and R. A. Borzooei, States on pseudo EQ-algebras, Soft Computing 26 (2022) 13219–13231.
[28] D. B. West, Introduction to graph theory, (Vol. 2), Upper Saddle River, N.J. 2001.
[29] R. P. Dilworth and M. Ward, Residuated lattices, Transactions of The American Society 45 (1939) 335–354.
[30] K. Blount and C. Tsinakis, The structure of residuated lattices, International Journal of Algebra and Computation 13 (2003) 437–461.
[31] L. C. Ciungu, Bosbach and Riečan states on residuated lattices, Journal of Applied Functional Analysis 3 (2008) 175–188.

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